CALCULUS DIFFERENTIATION GUIDE

I. Basic Differentiation Rules:

Let f, and g be differentiable functions of x with c a constant.

1. $\dfrac{d}{dx}(c) = 0$
2. $\dfrac{d}{dx}(cf(x)) = cf'(x)$
3. $\dfrac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$
4. $\dfrac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
5. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
6. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

II. Basic Differentiation Formulas:

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}u^n = nu^{n-1}\, u'$
2. $\dfrac{d}{dx}|u| = \dfrac{u}{|u|}u' = \dfrac{|u|}{u}\, u'$
3. $\dfrac{d}{dx}\sin(u) = \cos(u)\, u'$
4. $\dfrac{d}{dx}\cos(u) = -\sin(u)\, u'$
5. $\dfrac{d}{dx}\tan(u) = \sec^2(u)\, u'$
6. $\dfrac{d}{dx}\cot(u) = -\csc^2(u)\, u'$
7. $\dfrac{d}{dx}\sec(u) = \sec(u)\tan(u)\, u'$
8. $\dfrac{d}{dx}\csc(u) = -\csc(u)\cot(u)\, u'$
9. $\dfrac{d}{dx}\ln|u| = \dfrac{d}{dx}\ln(u) = \dfrac{1}{u}\, u'$
10. $\dfrac{d}{dx}\log_a|u| = \dfrac{d}{dx}\log_a(u) = \dfrac{1}{u\ln a}\, u'$
11. $\dfrac{d}{dx}e^u = e^u\, u'$
12. $\dfrac{d}{dx}a^u = a^u\,(\ln a)\, u'$
13. $\dfrac{d}{dx}\arcsin(u) = \dfrac{d}{dx}\sin^{-1}(u) = \dfrac{1}{\sqrt{1-u^2}}\, u'$
14. $\dfrac{d}{dx}\arccos(u) = \dfrac{d}{dx}\cos^{-1}(u) = -\dfrac{1}{\sqrt{1-u^2}}\, u'$
15. $\dfrac{d}{dx}\arctan(u) = \dfrac{d}{dx}\tan^{-1}(u) = \dfrac{1}{1+u^2}\, u'$
16. $\dfrac{d}{dx}\text{arccot}(u) = \dfrac{d}{dx}\cot^{-1}(u) = -\dfrac{1}{1+u^2}\, u'$
17. $\dfrac{d}{dx}\text{arcsec}(u) = \dfrac{d}{dx}\sec^{-1}(u) = \dfrac{1}{|u|\sqrt{u^2-1}}\, u'$
18. $\dfrac{d}{dx}\text{arccsc}(u) = \dfrac{d}{dx}\csc^{-1}(u) = -\dfrac{1}{|u|\sqrt{u^2-1}}\, u'$
19. $\dfrac{d}{dx}\sinh(u) = \cosh(u)\, u'$
20. $\dfrac{d}{dx}\cosh(u) = \sinh(u)\, u'$
21. $\dfrac{d}{dx}\tanh(u) = \text{sech}^2(u)\, u'$
22. $\dfrac{d}{dx}\coth(u) = -\text{csch}^2(u)\, u'$
23. $\dfrac{d}{dx}\text{sech}(u) = -\text{sech}(u)\tanh(u)\, u'$
24. $\dfrac{d}{dx}\text{csch}(u) = -\text{csch}(u)\coth(u)\, u'$

The following formulas are optional and are included for completeness:

1. $\dfrac{d}{dx}\text{arsinh}(u) = \dfrac{d}{dx}\sinh^{-1}(u) = \dfrac{1}{\sqrt{u^2+1}}\, u'$
2. $\dfrac{d}{dx}\text{arcosh}(u) = \dfrac{d}{dx}\cosh^{-1}(u) = \dfrac{1}{\sqrt{u^2-1}}\, u'$
3. $\dfrac{d}{dx}\text{artanh}(u) = \dfrac{d}{dx}\tanh^{-1}(u) = \dfrac{1}{1-u^2}\, u'$
4. $\dfrac{d}{dx}\text{arcoth}(u) = \dfrac{d}{dx}\coth^{-1}(u) = \dfrac{1}{1-u^2}\, u'$
5. $\dfrac{d}{dx}\text{arsech}(u) = \dfrac{d}{dx}\text{sech}^{-1}(u) = -\dfrac{1}{u\sqrt{1-u^2}}\, u'$
6. $\dfrac{d}{dx}\text{arcsch}(u) = \dfrac{d}{dx}\text{csch}^{-1}(u) = -\dfrac{1}{|u|\sqrt{1+u^2}}\, u'$

CALCULUS INTEGRATION GUIDE

I. **Basic Integration Rules:**

Let f and g be continuous functions of x with c a constant.

1. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$

2. $\displaystyle\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3. $\displaystyle\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, where we have substituted $u = g(x)$.

II. **Basic Integration Formulas:**

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1.$

2. $\displaystyle\int u^{-1}\,du = \int \frac{1}{u}\,du = \int \frac{du}{u} = \ln|u| + C$

3. $\displaystyle\int \sin(u)\,du = -\cos(u) + C$

4. $\displaystyle\int \cos(u)\,du = \sin(u) + C$

5. $\displaystyle\int \tan(u)\,du = \begin{cases} \ln|\sec(u)| + C \\ -\ln|\cos(u)| + C \end{cases}$

6. $\displaystyle\int \cot(u)\,du = \begin{cases} \ln|\sin(u)| + C \\ -\ln|\csc(u)| + C \end{cases}$

7. $\displaystyle\int \sec(u)\,du = \begin{cases} \ln|\sec(u) + \tan(u)| + C \\ -\ln|\sec(u) - \tan(u)| + C \end{cases}$

8. $\displaystyle\int \csc(u)\,du = \begin{cases} \ln|\csc(u) - \cot(u)| + C \\ -\ln|\csc(u) + \cot(u)| + C \end{cases}$

9. $\displaystyle\int \sec^2(u)\,du = \tan(u) + C$

10. $\displaystyle\int \csc^2(u)\,du = -\cot(u) + C$

11. $\displaystyle\int \sec(u)\tan(u)\,du = \sec(u) + C$

12. $\displaystyle\int \csc(u)\cot(u)\,du = -\csc(u) + C$

13. $\displaystyle\int e^u\,du = e^u + C$

14. $\displaystyle\int a^u\,du = \frac{1}{\ln a}\,a^u + C$

Let $a > 0$:

15. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$

16. $\displaystyle\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

17. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

III. **Techniques:**

1. Basic u-substitution: let u be the quantity in parentheses, the radicand, the denominator, or the exponent.

2. If the integrand is a rational function, divide if improper, otherwise split the numerator, resolve into partial fractions, or complete the square!

3. Integration by Parts: $\int u\,dv = uv - \int v\,du$. Rule of thumb: let $u =$ L(og.)I(nverse)A(lg.)T(rig.)E(xp.)

4. For trigonometric integrals:
   (a) Can you spare a cosine and convert the rest to sines?
   (b) Can you spare a sine and convert the rest to cosines?
   (c) Can you use a reduction formula?
      i. $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$.
      ii. $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$.
   (d) Can you spare a secant^2 and convert the rest to tangents ?
   (e) Can you spare a secant tangent and convert the rest to secants ?
   (f) If there are no secants, convert a tangent^2 to secant$^2 - 1$.
   (g) If you have an odd power of secant and no tangents, use parts by sparing a secant^2 for the dv.
   (h) To handle integrands with cosecants and cotangents, mimic the above strategies for secants and tangents.
   (i) If all else fails, try using a Pythagorean Conjugate, converting back to sines and cosines, or the infamous $z = \tan\left(\frac{\theta}{2}\right)$ substitution.

5. For integrands involving $\sqrt{a^2 - u^2}$, use the trig. sub. $u = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

6. For integrands involving $\sqrt{u^2 - a^2}$, use the trig. sub. $u = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$.

7. For integrands involving $\sqrt{u^2 + a^2}$, use the trig. sub. $u = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

MATH 2500 INTEGRATION PRACTICE

Name:____

Find the following integrals.

1. $\displaystyle\int \sqrt{1-x}\,dx$

2. $\displaystyle\int x\sqrt{1-x^2}\,dx$

3. $\displaystyle\int x\sqrt{1-x}\,dx$

4. $\displaystyle\int \sin(x)\sqrt{1-\cos(x)}\,dx$

5. $\displaystyle\int \frac{3}{(x-1)^3}\,dx$

6. $\displaystyle\int \frac{3x}{(x-1)^5}\,dx$

7. $\displaystyle\int \frac{3x}{(x^2-1)^2}\,dx$

8. $\displaystyle\int \frac{3}{x^2-2x+1}\,dx$

   HINT: Factor...

9. $\displaystyle\int \frac{\sec(x)}{\tan^2(x)+1}\,dx$

   HINT: Identities...

10. $\displaystyle\int_0^{2\pi} \sqrt{1-\cos(x)}\,dx$

    HINT: $1-\cos(x) = 2\sin^2(x/2)\ \ldots$

ANSWERS:

1. $\int \sqrt{1-x}\,dx = -\frac{2}{3}(1-x)^{3/2} + C$

2. $\int x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2} + C$

3. $\int x\sqrt{1-x}\,dx = \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C$

4. $\int \sin(x)\sqrt{1-\cos(x)}\,dx = \frac{2}{3}(1-\cos(x))^{3/2} + C$

5. $\int \frac{3}{(x-1)^3}\,dx = -\frac{3}{2}(x-1)^{-2} + C$

6. $\int \frac{3x}{(x-1)^5}\,dx = -(x-1)^{-3} - \frac{3}{4}(x-1)^{-4} + C$

7. $\int \frac{3x}{(x^2-1)^2}\,dx = -\frac{3}{2}(x^2-1)^{-1} + C$

8. $\int \frac{3}{x^2-2x+1}\,dx = \int \frac{3}{(x-1)^2}\,dx = -3\,(x-1)^{-1} + C$

9. $\int \frac{\sec(x)}{\tan^2(x)+1}\,dx = \int \frac{\sec(x)}{\sec^2(x)}\,dx = \int \cos(x)\,dx = \sin(x) + C$

10. $\int_0^{2\pi} \sqrt{1-\cos(x)}\,dx = \int_0^{2\pi} \sqrt{2\sin^2(x/2)}\,dx = \ldots = 4\sqrt{2}$

MATH 2600: INTEGRATION REVIEW

Find the following integrals. Yep, you guessed it, check your answer by taking the derivative!

1. $\int \sqrt{4-3x}\, dx$
2. $\int x\sqrt{4-3x}\, dx$
3. $\int \frac{x}{\sqrt{4-9x^2}}\, dx$
4. $\int \frac{1}{\sqrt{4-9x^2}}\, dx$
5. $\int \frac{x^3}{\sqrt{9x^2-4}}\, dx$
6. $\int \frac{1}{x\sqrt{9x^2-4}}\, dx$
7. $\int \frac{x}{9x^2+4}\, dx$
8. $\int \frac{1}{9x^2+4}\, dx$
9. $\int \frac{1}{x^2-6x+9}\, dx$
10. $\int \frac{1}{x^2-6x+10}\, dx$
11. $\int \frac{1}{x+\sqrt{x}}\, dx$
12. $\int \frac{1}{\sqrt{x}+x\sqrt{x}}\, dx$
13. $\int \frac{e^x+1}{e^x}\, dx$
14. $\int \frac{e^x}{1+e^x}\, dx$
15. $\int \frac{e^x}{1+e^{2x}}\, dx$
16. $\int \frac{e^x}{\sqrt{1-e^x}}\, dx$
17. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$
18. $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx$
19. $\int \frac{\sin(x)+1}{\cos(x)}\, dx$
20. $\int \frac{\cos(x)}{\sin(x)+1}\, dx$
21. $\int \frac{1}{\sin(x)+1}\, dx$
22. $\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\, dx$
23. $\int \frac{\sqrt{1+\ln(x)}}{x}\, dx$
24. $\int \frac{2^x}{\sqrt{9-4^x}}\, dx$
25. $\int \frac{e^{\arctan(x)}}{x^2+1}\, dx$
26. $\int \frac{1}{x\ln(x)\sqrt{(\ln(x))^2-1}}\, dx$
27. $\int 3e^{2\ln(x)}\, dx$
28. Find $\int \frac{2x^3-4x^2+10x-1}{x^2-2x+5}\, dx$
29. Find $\int \frac{1}{x\sqrt{x-1}}\, dx$ two ways: using $u=\sqrt{x-1}$ and $u=\sqrt{x}$.
30. Evaluate $\int_0^{\pi/2} \sqrt{1-\cos(\theta)}\, d\theta$

CHALLENGE: Find $\int_0^1 \frac{x^4(1-x)^4}{x^2+1}\, dx$

HINTS:

1. Power rule
2. $u = 4 - 3x$, $x = \dfrac{4-u}{3}$
3. $u = 4 - 9x^2$
4. arcsine form
5. $u = 9x^2 - 4$, $x^2 = \dfrac{u+4}{9}$
6. arcsecant form
7. $u = 9x^2 + 4$
8. arctangent form
9. Factor; power rule
10. Complete the square; arctangent form
11. Factor $\sqrt{x}$ from denominator; $u = 1 + \sqrt{x}$.
12. Factor $\sqrt{x}$ from denominator; $u = \sqrt{x}$; arctangent form
13. separate numerator / bring up e^x from denominator as e^{-x} and distribute
14. $u = e^x + 1$
15. $u = e^x$; arctangent form
16. $u = 1 - e^x$
17. $u = e^x$; arcsine form
18. $u = e^x$; arcsecant form (multiply numerator and denominator by e^x)
19. Separate numerator / bring $\cos(x)$ up as $\sec(x)$ / rewrite integrand as $\tan(x) + \sec(x)$.
20. $u = \sin(x) + 1$
21. Multiply numerator and denominator by conjugate: $1 - \sin(x)$ and separate numerator.
22. $u = \arcsin(x)$
23. $u = 1 + \ln(x)$
24. $u = 2^x$; arcsine form
25. $u = \arctan(x)$
26. $u = \ln(x)$; arcsecant form
27. use properties of logs and exponents to reduce integrand to $3x^2$.
28. long division then complete the square.
29. $u = \sqrt{x-1}$ results in an arctangent form; $u = \sqrt{x}$ results in an arcsecant form.
30. $2\sqrt{2} - 2$

MATH 2600 TECHNIQUES OF INTEGRATION PRACTICE

NAME: ______________________________

1. $\int \arcsin(x)\,dx$
2. $\int x^2 \ln(x)\,dx$
3. $\int x \sin(x^2)\,dx$
4. $\int x^2 \sin(x)\,dx$
5. $\int x^2 \sqrt{9 - x^2}\,dx$
6. $\int x^2 \sqrt{x^2 - 9}\,dx$
7. $\int \frac{x}{x^2 - 9}\,dx$
8. $\int \frac{1}{x^2 - 9}\,dx$
9. $\int e^{2x} \sqrt{1 + e^{2x}}\,dx$
10. $\int e^{x} \sqrt{1 + e^{2x}}\,dx$
11. $\int \sqrt{1 + e^{2x}}\,dx$
12. $\int \frac{\sqrt{9 + x^2}}{x}\,dx$

HINTS!

1. Integration by parts: let $u = \arcsin(x)$; $dv = dx$
2. Integration by parts: let $u = \ln(x)$; $dv = x^2\,dx$
3. u-substitution: let $u = x^2$
4. Tabular integration with $u = x^2$ and $dv = \sin(x)\,dx$
5. Trigonometric substitution: let $x = 3\sin(\theta)$
6. Trigonometric substitution: let $x = 3\sec(\theta)$
7. u-substitution: let $u = x^2 - 9$
8. Partial fractions.
9. u-substitution: let $u = 1 + e^{2x}$
10. u-substitution followed by trigonometric substitution: let $u = e^x$ and then let $u = \tan(\theta)$.
11. u-substitution: let $u = \sqrt{1 + e^{2x}}$
12. Trigonometric substitution: let $x = 3\tan(\theta)$